Mean Field Games in Macroeconomics

Matias Bayas-Erazo

University of Zürich

Motivation

- Modern macro needs tractable GE models with aggregate and idiosyncratic uncertainty
- Given individual state $x_i \in \mathcal{X}$ and the distribution of agents $g \in \mathcal{P}_2(\mathcal{X})$, solve

$$V(x_i, g) = \max_{\alpha \in \mathcal{A}} \left\{ u(\alpha, x_i, g) + \beta \mathbb{E}^{i,g} \left[V(x_i', g') \right] \right\} \quad s.t \quad x_i' \in \Gamma_x(\alpha, x_i, g)$$

- Distribution evolves as $g' \in \Gamma_g(g)$ — still an ∞-dimensional object
- Krusell and Smith (1998): assume approximate aggregation and solve numerically
- **Can mean field game theory do better?**

Mean field games — background

- Formalized by Lasry and Lions (2006): limit of N-player games as $N \to \infty$
- Framework for systems with many rational, symmetric players
- **Key insight**: equilibrium reduces to a system of PDEs
    1. Idiosyncratic shocks: reasonably well understood
    2. Common noise: much less is known
- Achdou et al. (2022): heterogeneous agent models can be cast in this form

The MFG system

Two equations at the core:

1. **Hamilton-Jacobi-Bellman** — optimal behavior of agents
    - Solved backward: given the future, what is optimal today?

2. **Kolmogorov Forward** — evolution of the density of agents
    - Solved forward: given optimal behavior, how does the distribution evolve?

Finding an equilibrium $=$ finding a fixed point of the coupled HJB–KF system.

Standard MFG: Bewley-Aiyagari in continuous time

The household problem

- Consumer with income process $\{y_t\}$ chooses consumption $\{c_t\}$ to maximize

$$\mathbb{E}\left[\int_0^\infty e^{-\rho t} u(c_t) dt\right]$$

subject to

$$da_t = [ra_t + y_t - c_t]\, dt, \quad a_t \geq -\phi$$

- Income y follows a two-state Poisson process with intensities λ_1, λ_2
- Consumer takes the interest rate r as given

Stationary equilibrium as an MFG system

1. **HJB** + state constraint boundary condition (SCBC):

$$0 = \sup_c \left\{ -\rho v_j(a) + u(c) + v_j'(a)(ra + y_j - c) + \lambda_j (v_{-j}(a) - v_j(a)) \right\}$$

$$v_j'(-\phi) \geq u'(-r\phi + y_j), \quad j = 1, 2$$

2. **Kolmogorov Forward**:

$$0 = -\frac{d}{da}\left[s_j(a)\, g_j(a) \right] - \lambda_j g_j(a) + \lambda_i g_i(a)$$

3. **Market clearing**:

$$\int_{-\phi}^{\infty} a \left[g_1(a) + g_2(a) \right] da = 0$$

Understanding the state constraint

Following Soner (1986): assume a continuous optimal policy $c^*(a)$ and differentiable v.

1. The borrowing constraint requires $r\underline{a} + y_j - c^*(\underline{a}) \geq 0$

2. Optimality of c^* gives $H_j(a, v'(a)) = u(c^*) + v'(a)(ra + y_j - c^*)$

3. For any $\gamma \geq 0$:

$$H_j(\underline{a}, v'(\underline{a}) + \gamma) \geq H_j(\underline{a}, v'(\underline{a})) + \gamma(r\underline{a} + y_j - c^*(\underline{a})) \geq H_j(\underline{a}, v'(\underline{a}))$$

4. Convexity of H in p yields the equivalent condition

$$\partial_p H_j(\underline{a}, v'(\underline{a})) \geq 0$$

Intuition for the Kolmogorov Forward equation

- With time dependence:

$$\frac{\partial g_j(a,t)}{\partial t} = -\frac{\partial}{\partial a}\left[s_j(a)g_j(a,t)\right] - \lambda_j g_j(a,t) + \lambda_i g_i(a,t)$$

- Poisson terms (λ_j, λ_i): mass flows between income states
- Transport term: if $s > 0$ at a, mass flows rightward
  - Gain from $s_j(a-\epsilon)g_j(a-\epsilon,t)$, lose $s_j(a)g_j(a,t)$
  - Same expression regardless of sign of s — hence the divergence form

Application: impulsive MFG with lumpy investment

Firms with lumpy capital and convex labor adjustment

- Firms produce $y_t = f(k_t, l_t)$, taking r and w as given
- Without adjustment, capital depreciates:

$$dk_t = -\delta k_t \, dt + \sigma \, dB_t$$

- Capital adjustment is lumpy: jumping from k to $k + i$ costs

$$\mathcal{G}(i) = F + p^+ i^+ + p^- i^-$$

- Labor adjustment is convex: hiring at rate n_t costs $\mathcal{C}(n_t, l_t)$

$$dl_t = [n_t - \rho l_t] \, dt$$

- This gives a **combined stochastic control and impulse control** problem

The HJBVI equation

- Define the intervention operator:

$$\mathcal{M}v(k,l) := \sup_i \{v(k+i,l) - \mathcal{G}(i)\}$$

- The value function solves the **Hamilton-Jacobi-Bellman Variational Inequality**:

$$0 = \max\left\{\sup_n \left\{-rv + \pi(k,l,n) + \partial_l v\,[n - \rho l] + \partial_k v\,[-\delta k] + \frac{\sigma^2}{2}\partial_{kk} v\right\}, \mathcal{M}v - v\right\}$$

- Interpretation: optimal stopping with endogenous stopping value

- Extends standard MFG to settings with both continuous and discrete adjustments

Numerical strategy

Solve as a sequence of optimal stopping problems:

1. Compute no-intervention value v^0 from standard HJB:

$$0 = \sup_n \left\{ -rv^0 + \pi + \partial_l v^0 [n - \rho l] + \partial_k v^0 [-\delta k] + \frac{\sigma^2}{2} \partial_{kk} v^0 \right\}$$

2. Define v^j inductively as the solution to control + stopping:

$$0 = \max \left\{ \sup_n \{\cdots\}, \mathcal{M} v^{j-1} - v^j \right\}$$

3. **Convergence**: $\lim_{j\to\infty} v^j(k,l) = v(k,l)$ [Øksendal and Sulem 2007]

Implementation: finite differences + upwind scheme for HJB; linear complementarity for stopping.

Inaction region

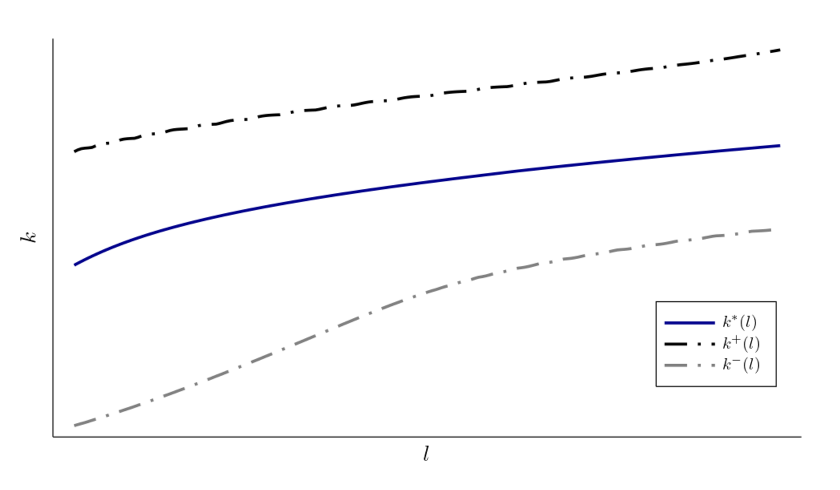

Figure 1: Inaction region in the (k, l) space — firms adjust capital only outside this region

Inaction region with irreversibility

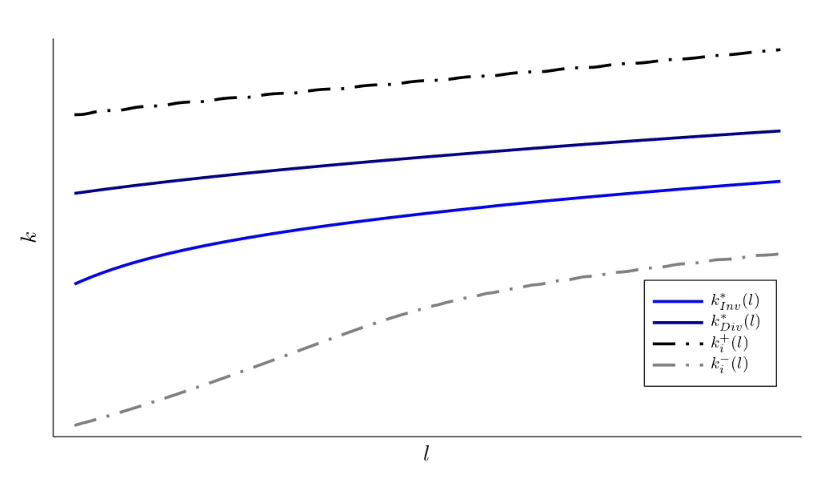

Figure 2: Asymmetric adjustment costs shift the inaction region

Equilibrium distributions

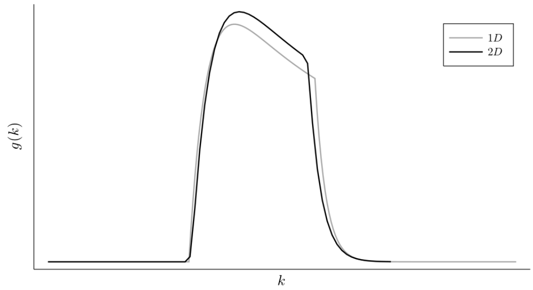

Figure 3: Capital distribution

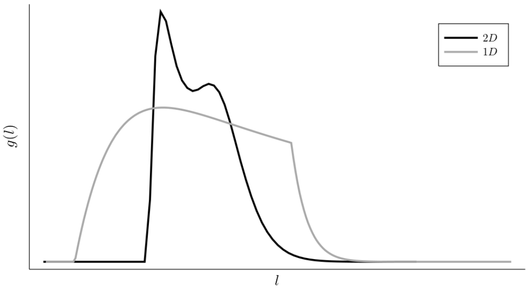

Figure 4: Labour distribution

Joint distribution

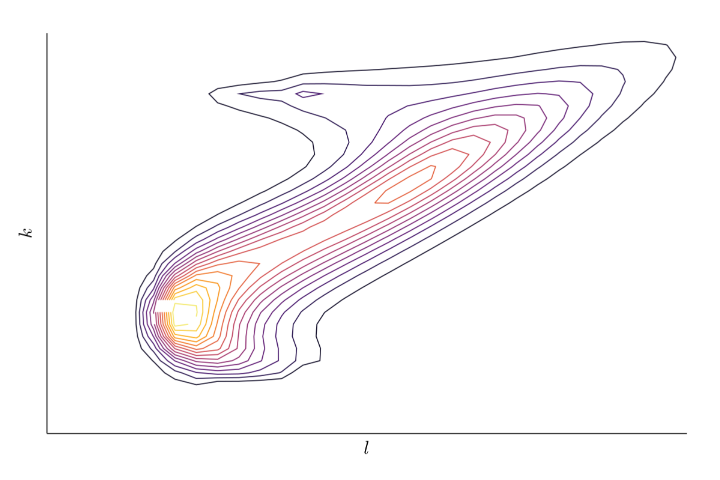

Figure 5: Joint distribution in the baseline model

General equilibrium

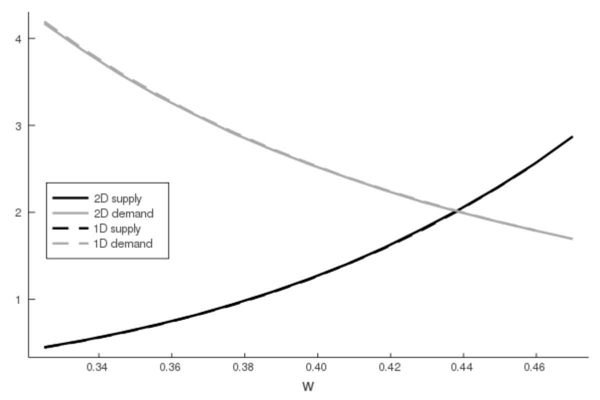

Figure 6: Equilibrium in the labour market

Outlook

Open directions

- **Aggregate shocks** $\implies$ the Master equation
  - Bilal (2023): perturbation of the Master equation (FAME) reduces to a standard Bellman equation
- **Beyond the continuum**: MFG with major and minor players
- **Comparative advantage of MFG**: when GE enters individual problems beyond a small set of prices and the Krusell and Smith (1998) strategy breaks down
- **Machine learning for MFG**: neural networks as solvers for HJB–KF systems
  - Ruthotto et al. (2020): deep learning framework for high-dimensional MFG and MFC
  - Han et al. (2021): DeepHAM — global solution for HA models via neural nets
  - Payne et al. (2024): deep learning to solve Extended Master Equations in search models

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